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Fatou Type Theorems for Series in Mittag-Leffler Functions

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Abstract. In studying the behaviour of series, defined by means of the Mittag-Leffler functions, on the boundary of its domain of convergence in the complex plane, we give analogues of the classical theorems for the power series like Cauchy-Hadamard, Abel, as well as Fatou theorems. The asymptotic formulae for the Mittag-Leffler functions in the cases of "large" values of indices that are used in the proofs of the convergence theorems for the considered series are also provided.

Keywords: Mittag-Leffler functions, inequalities, asymptotic formula, Cauchy-Hadamard, Abel and Fatou type theorems.
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1. INTRODUCTION

The Mittag-Leffler functions E_α (Mittag-Leffler, 1902-1905) and $E_{\alpha,\beta}$ (Agarwal 1953, see also [3]), are defined in the whole complex plane $\mathbb{C}$ by the power series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \; E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \, \beta > 0. \tag{1.1}$$

The Mittag-Leffler (M-L) functions (1.1) are examples of entire functions of a given order $\rho = 1/\alpha$ and a type $\sigma = 1$. They have been studied in details by Dzrbashjan [1]: asymptotic formulae in different parts of the complex plane, distribution of the zeros, kernel functions of inverse Borel type integral transforms, various relations and representations. The detailed properties of these functions can be found in the contemporary monographs of Kilbas et al. [2] and Podlubny [14]. The M-L function and its generalizations appear as solutions of fractional order differential and integral equations. They find also many applications in Statistics (as the so-called M-L density), see e.g. [8].

Recently, series in Mittag-Leffler functions of the kind (1.1) have been considered. Studying their behaviour on the boundary of the domain of convergence, Cauchy-Hadamard, Abel and Tauberian type theorems have been proved. In this paper we give some of these results and prove a theorem of Fatou type. Such kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler functions (as for example in Kiryakova [6] and Sandev, Tomovski and Dubbeldam [17]).

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2. INEQUALITIES AND ASYMPTOTIC FORMULAE

First of all, denoting

$$\theta_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kn+1)}, \quad \theta_{n,\beta}(z) = \Gamma(\beta) \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kn+\beta)}, \quad \theta_{\alpha,n}(z) = \Gamma(n) \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k+n)},$$

we give some inequalities and asymptotic formulae for "large" values of indices as follows (for details see [13]).

Lemma 2.1. *Let $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists a constant $\widetilde{C}$, $0 < \widetilde{C} < \infty$, such that*

$$|\theta_n(z)| \leq \widetilde{C}/n!, \quad |\theta_{n,\beta}(z)| \leq \widetilde{C}/(n-1)!, \quad |\theta_{\alpha,n}(z)| \leq \widetilde{C} \frac{\Gamma(n)}{\Gamma(\alpha+n)}, \tag{2.1}$$

for all the natural numbers n and each $z \in K$.

Theorem 2.1. *For the Mittag-Leffler functions E_n, $E_{n,\beta}$, $E_{\alpha,n}$ $(n \in \mathbb{N})$, the following asymptotic formulae*

$$E_n(z) = 1 + \theta_n(z), \;\; z \in \mathbb{C}, \quad \theta_n(z) \to 0 \quad as \;\; n \to \infty \tag{2.2}$$

$$E_{n,\beta}(z) = \frac{1}{\Gamma(\beta)} \, (1 + \theta_{n,\beta}(z)), \;\; z \in \mathbb{C}, \quad \theta_{n,\beta}(z) \to 0 \quad as \; n \to \infty \tag{2.3}$$

$$E_{\alpha,n}(z) = \frac{1}{\Gamma(n)} \, (1 + \theta_{\alpha,n}(z)), \;\; z \in \mathbb{C}, \quad \theta_{\alpha,n}(z) \to 0 \quad as \; n \to \infty \tag{2.4}$$

are valid. The functions $\theta_n(z)$, $\theta_{n,\beta}(z)$, $\theta_{\alpha,n}(z)$ are holomorphic for $z \in \mathbb{C}$. The convergence is uniform on the compact subsets of the complex plane $\mathbb{C}$.

Note 2.1. According to the asymptotic formulae (2.2) - (2.4), it follows there exists a natural number N_0 such that the functions E_n, $\Gamma(n)E_{\alpha,n}$, $\Gamma(\beta)E_{n,\beta}$ have no zeros for $n > N_0$.

3. SERIES IN MITTAG-LEFFLER FUNCTIONS. THEOREMS OF CAUCHY-HADAMARD AND ABEL TYPE

We introduce auxiliary functions, related to Mittag-Leffler's functions, adding $\widetilde{E}_0(z)$, $\widetilde{E}_{0,\beta}(z))$ and $\widetilde{E}_{\alpha,0}(z)$ just for completeness, namely:

$$\widetilde{E}_0(z) = 1; \quad \widetilde{E}_n(z) = z^n E_n(z), \; n \in \mathbb{N},$$

$$\widetilde{E}_{0,\beta}(z) = 1; \quad \widetilde{E}_{n,\beta}(z) = \Gamma(\beta) z^n E_{n,\beta}(z), \; n \in \mathbb{N}; \quad \beta > 0,$$

$$\widetilde{E}_{\alpha,0}(z) = 1; \quad \widetilde{E}_{\alpha,n}(z) = \Gamma(n) z^n E_{\alpha,n}(z), \; n \in \mathbb{N}; \quad \alpha > 0, \tag{3.1}$$

and consider the series in these functions, respectively:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}_n(z), \qquad \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\beta}(z), \qquad \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,n}(z), \tag{3.2}$$

with complex coefficients a_n $(n = 0, 1, 2, \ldots)$.

We give some previous results for series of the kind (3.2) and consider their behaviour on an arc of the unit circle $|z| = 1$, all the points of which (including the ends) are regular to the sum of the series. Similar convergence theorems have been proved also for series in other special functions, for example, for series in Laguerre and Hermite polynomials [15] - [16], and resp. by the author [9] - [12] for systems of Bessel functions, their fractional indices analogues (as Bessel-Maitalnd functions) and also of multi-index Mittag-Leffler functions (in the sense of [3],[4],[5]).

In the beginning we give a theorem of Cauchy-Hadamard type for each of the above series.

Theorem 3.1. (of Cauchy-Hadamard type). *The domain of convergence of each of the series (3.2) with complex coefficients a_n is the disk $|z| < R$ with a radius of convergence $R = 1/\Lambda$, where*

$$\Lambda = \limsup_{n\to\infty} \left(\, |a_n| \, \right)^{1/n}. \tag{3.3}$$

More precisely, the series (3.2) are absolutely convergent on the disk $|z| < R$ and divergent on the domain $|z| > R$. The cases $\Lambda = 0$ and $\Lambda = \infty$ can be included in the general case, provided $1/\Lambda$ means ∞, respectively 0.

Corollary 3.1.1. *Let any of the series (3.2) converges at the point $z_0 \neq 0$. Then it is absolutely convergent on the disk $D = \{z : |z| < |z_0|, \ z \in \mathbb{C}\}$. Inside of the disk $|z| < 1/\Lambda = R$, i.e. on each closed disk $|z| \leq r < R$ (Λ defined by (3.3)), the convergence is uniform.*

The very disk of convergence is not obligatory a domain of uniform convergence. Moreover, on its boundary the series may even be divergent.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0 and d_φ be the part of the angular domain g_φ, closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle.

The following theorem is valid.

Theorem 3.2. (of Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, Λ be defined by (3.3), $0 < \Lambda < \infty$. Let $K = \{z : z \in \mathbb{C}, |z| < R, R = 1/\Lambda\}$. If $f(z)$, $g(z;\beta)$, $h(z;\alpha)$ are the sums respectively of the first, second and third of the series (3.2) on the domain K, and these series converge at the point z_0 of the boundary of K, then the series (3.2) are uniformly convergent on the domain d_φ. and*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z_0), \ \lim_{z\to z_0} g(z;\,\beta) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\,\beta}(z_0), \ \lim_{z\to z_0} h(z;\,\alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\,n}(z_0),$$

provided $|z| < R$ and $z \in g_\varphi$.

Note 3.1. If some of the series (3.2) has a finite and non-zero radius of convergence R, it converges at the point $z_0 \in C(0,R)$ and F is the holomorphic function defined by this series in its domain of convergence, then by Theorem 3.2. it follows that

$$\lim_{z \to z_0,\, z \in d_\varphi} F(z) = F(z_0),$$

i.e. the restriction of the function F to each set of the kind d_φ is continuous at the point z_0.

Note 3.2. Note that each of the functions $\widetilde{E}_n(z), \widetilde{E}_{n,\beta}(z), \widetilde{E}_{\alpha,n}(z)$, $(n \in \mathbb{N})$, being an entire function, no identically zero, has no more than a finite number of zeros in the closed and bounded set $|z| \leq R$ ([7], vol.1, ch. 3, §6, 6.1, p.305). Moreover, because of Note 2.1., no more than a finite number of these functions have some zeros.

4. FATOU TYPE THEOREMS

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $\limsup\limits_{n\to\infty} (\,|a_n|\,)^{1/n} = R$, and $\widetilde{f}(z)$ be the sum of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ on the open disk $U(0;R) = \{z : z \in \mathbb{C}, |z| < R\}$, i.e.

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U(0;R).$$

A point $z_0 \in \partial U(0;R)$ is called *regular* for the function $\widetilde{f}$ if there exist a neighbourhood $U(z_0;\rho)$ and a function $\widetilde{f}^*_{z_0} \in \mathscr{H}(U(z_0;\rho))$ (the space of complex-valued functions, holomorphic in the set $U(z_0;\rho)$), such that $\widetilde{f}^*_{z_0}(z) = \widetilde{f}(z)$ for $z \in U(z_0;\rho) \cap U(0;R)$. By this definition it follows that the set of regular points of the power series is an open subset of the circle $C(0;R) = \partial U(0;R)$ with respect to the relative topology on $\partial U(0;R)$, i.e. the topology induced by that of $\mathbb{C}$.

In general, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum of such points.

For example, the power series $\sum\limits_{n=0}^{\infty} z^n$ is divergent at each point of the circle $C(0;1)$ regardless of the fact that all the points of this circle, except $z = 1$, are regular for its sum. The series $\sum\limits_{n=1}^{\infty} n^{-2} z^n$ is (absolutely) convergent at each point of the circle $C(0;1)$, but nevertheless one of them, namely $z = 1$, is a singular (i.e. not regular) for its sum.

But under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation does exist. More precisely, the following proposition holds ([7], vol.1, ch. 3, §7, 7.3, p.357).

Theorem 4.1. (of Fatou). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, such that*

$$\limsup_{n\to\infty} (\,|a_n|\,)^{1/n} = 1, \tag{4.1}$$

and $\widetilde{f}(z)$ be the sum of the power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{4.2}$$

on the unit disk $D = \{z : z \in \mathbb{C}, |z| < 1\}$, i.e. $\widetilde{f}(z) = \sum_{n=0}^{\infty} a_n z^n$, for $z \in D$. Let σ be an arbitrary arc of the unit circle $C(0;1)$ with all its points (including the ends) regular to the function $\widetilde{f}$. Let $\lim_{n\to\infty} a_n = 0$. Then the power series (4.2) converges, even uniformly, on the arc σ.

Let us point out that under the hypothesis of the above assertion there exists a region $G \supset \sigma$ and a function $\widetilde{f}^* \in \mathscr{H}(G)$ such that $\widetilde{f}^*(z) = \widetilde{f}(z)$ for $z \in G \cap D$.

It means that the function $\widetilde{f}^*$ is an analytical continuation of the function $\widetilde{f}$ outside of the disk D. Moreover, as it is not difficult to see, the series (4.2) converges on that open arc $\widetilde{\sigma} \subset C(0,1)$ which contains σ and is included in the region G. Then Abel's theorem yields that the sum of the series (4.2) is $\widetilde{f}(z)$ for each $z \in \widetilde{\sigma}$. Therefore, we may assume that the power series (4.2) represents the function $\widetilde{f}$ even on the arc $\widetilde{\sigma}$.

Proposition like Theorem 4.1 holds also for series in the Laguerre and Hermite systems (see e.g. [16]). Here we give such a type of theorem for series in Mittag-Leffler functions, as follows.

Theorem 4.2. (of Fatou type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the condition (4.1) and $f(z)$, $g(z;\beta)$, $h(z;\alpha)$ be the sums respectively of the first, second and third of the series (3.2) on the disk $D = \{z : z \in \mathbb{C}, |z| < 1\}$, i.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z), \;\; g(z;\,\beta) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\,\beta}(z), \;\; h(z;\,\alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\,n}(z); \;\; z \in D. \quad (4.3)$$

Let σ be an arbitrary arc of the unit circle $|z| = 1$ with all its points (including the ends) regular to the function f (resp. g or h). Let $\lim_{n\to\infty} a_n = 0$ and $\widetilde{E}_n(z) \neq 0$ (respectively $\widetilde{E}_{n,\,\beta}(z) \neq 0$, $\widetilde{E}_{\alpha,\,n}(z) \neq 0$) for $z \in \sigma$. Then the first (resp. second or third) of the series (3.2) converges, even uniformly, on the arc σ.

Proof. Here, we give the proof for the third of the series (3.2). Since all the points of the arc σ are regular to the function $h(z;\alpha)$ there exists a region $G \supset \sigma$ where the function h can be continued. Denoting $\widetilde{G} = G \cup D$, we define the function ψ in the region $\widetilde{G}$ by the equality

$$\psi(z) = h(z;\alpha), \quad z \in D.$$

More precisely, it means that h has a single valued analytical continuation in $\widetilde{G}$.

Let $\rho > 0$ be the distance between the boundary $\partial\widetilde{G}$ of the region $\widetilde{G}$ and the arc σ ($\partial\widetilde{G}$ contains a part of the unit circle $|z| = 1$), and take the points ζ_1, $\zeta_2 \notin \sigma$, $|\zeta_1| = |\zeta_2| = 1$, such that the distances between each of the points ζ_1, ζ_2 and the respective closer end of the arc σ are equal to $\rho/2$, and $z_1 = \zeta_1(1+\rho/2)$, $z_2 = \zeta_2(1+\rho/2)$.

Define

$$\varphi_n(z) = \psi(z) - \sum_{k=0}^{n} a_k \widetilde{E}_{\alpha,k}(z), \quad \omega_n(z) = \frac{\varphi_n(z)}{\widetilde{E}_{\alpha,n+1}(z)}(z-\zeta_1)(z-\zeta_2). \quad (4.4)$$

In order to prove that the sequence $\left\{\sum_{k=0}^{n} a_k \widetilde{E}_{\alpha,k}(z)\right\}$ is uniformly convergent on the arc σ, it is sufficiently to show that the sequence $\{\omega_n(z)\}_{n=0}^{\infty}$ tends uniformly to zero on the boundary $\partial\Delta$ of the sector $\Delta = Oz_1z_2$ which is a compact set.

To this end, we come back to (2.1). Just mention that since $\lim\limits_{n\to\infty}\frac{\Gamma(n)}{\Gamma(\alpha+n)} = \lim\limits_{n\to\infty}\frac{1}{n^\alpha} = 0$, there exist numbers C and $\widetilde{N}$ such that $|1+\theta_{\alpha,n}(z)| \le C/2$ for all the values of $n \in \mathbb{N}$ and $1/2 \le |1+\theta_{\alpha,n}(z)| \le 2$ for $n > \widetilde{N}$ on an arbitrary compact subset of $\mathbb{C}$.

Now, taking $\varepsilon > 0$ and setting

$$R = 1+\rho/2, \quad \varepsilon_1 = \frac{\varepsilon\rho^3}{8(8CR^2+\rho)}, \quad M = \max_{z\in[\Delta]}|\psi(z)| \qquad ([\Delta] = \Delta\cup\partial\Delta),$$

we have to consider four cases as follows.

1) First, let $z \in (O,\zeta_1)\cup(O,\zeta_2) \subset D$.

In the unit disk, according to (4.4) and (3.1), we have consecutively:

$$\omega_n(z) = \sum_{k=1}^{\infty} a_{n+k}\frac{\widetilde{E}_{\alpha,n+k}(z)}{\widetilde{E}_{\alpha,n+1}(z)}(z-\zeta_1)(z-\zeta_2),$$

$$\omega_n(z) = \sum_{k=n+1}^{\infty} a_k z^{k-n-1}\frac{(1+\theta_{\alpha,k}(z))}{(1+\theta_{\alpha,n+1}(z))}(z-\zeta_1)(z-\zeta_2). \tag{4.5}$$

Since $a_n \to 0$, there exists a number $N_1 = N_1(\varepsilon_1) > \widetilde{N}$, such that

$$|\omega_n(z)| \le \varepsilon_1 \sum_{k=n+1}^{\infty} |z|^{k-n-1}\left|\frac{(1+\theta_{\alpha,k}(z))}{(1+\theta_{\alpha,n+1}(z))}\right| |(z-\zeta_1)||(z-\zeta_2)|$$

$$< 2C\varepsilon_1 \sum_{k=n+1}^{\infty} |z|^{k-n-1}(1-|z|) = 2C\varepsilon_1$$

for $n > N_1$, i.e.

$$|\omega_n(z)| < 2C\varepsilon_1. \tag{4.6}$$

2) $z \in (\zeta_1, z_1)\cup(\zeta_2, z_2)$.

In this case $|z-\zeta_1| = |z|-1$, $|z-\zeta_2| \le |z|+|\zeta_2| < 2R$, and taking into account (2.4), (3.1) and (4.4) we can write the following inequalities for the absolute value of $\omega_n(z)$

$$\omega_n(z) = \frac{\psi(z) - \sum\limits_{k=0}^{n} a_k z^k(1+\theta_{\alpha,k}(z))}{z^{n+1}(1+\theta_{\alpha,n+1}(z))}(z-\zeta_1)(z-\zeta_2),$$

namely

$$|\omega_n(z)| \le \frac{M + \sum\limits_{k=0}^{n} |a_k||z|^k|(1+\theta_{\alpha,k}(z))|}{|z|^{n+1}|(1+\theta_{\alpha,n+1}(z))|}\; 2R(|z|-1)$$

$$< 2R\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right)\frac{(|z|-1)}{|z|^{n+1}} + 2\varepsilon_1 RC\,\frac{(|z|-1)}{|z|^{n+1}}\sum_{k=N_1+1}^{n}|z|^k\,.$$

Furthermore, having in mind that

$$\frac{(|z|-1)}{|z|^{n+1}} < \frac{(|z|-1)}{|z|^{n+1}-1} = \frac{1}{|z|^n+\cdots+1} < \frac{1}{n+1}, \quad \sum_{k=N_1+1}^{n}|z|^k = \frac{|z|^{n+1}-|z|^{N_1+1}}{(|z|-1)} < \frac{|z|^{n+1}}{(|z|-1)},$$

we conclude that

$$|\omega_n(z)| < \frac{2R}{n+1}\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right) + 2\varepsilon_1 RC.$$

Then, since $n^{-1} \to 0$, there exists a number $N_2 = N_2(\varepsilon_1) > N_1$ such that

$$\frac{2R}{n+1}\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right) < \varepsilon_1$$

for $n > N_2$, i.e.

$$|\omega_n(z)| < (1+2RC)\varepsilon_1. \tag{4.7}$$

3) z belongs to the arc $\widehat{z_1 z_2}$ (including the ends).

Then $|z-\zeta_1| < 2R$, $|z-\zeta_2| < 2R$ and hence

$$|\omega_n(z)| < \frac{4R^2\left(2M + \sum\limits_{k=0}^{n} C|a_k|R^k\right)}{R^{n+1}} < \frac{4\left(2M + \sum\limits_{k=0}^{N_1} C|a_k|R^k\right)}{R^{n-1}} + \frac{4\varepsilon_1 C\left(\sum\limits_{k=N_1+1}^{n} R^k\right)}{R^{n-1}}$$

$$< \frac{4\left(2M + \sum\limits_{k=0}^{N_1} C|a_k|R^k\right)}{R^{n-1}} + \frac{8\varepsilon_1 CR^2}{\rho}.$$

Since $R^{-n} \to 0$, there exists a number $N_3 = N_3(\varepsilon_1) > N_1$, such that

$$|\omega_n(z)| < \left(\frac{8CR^2}{\rho} + 1\right)\varepsilon_1 \tag{4.8}$$

for $n > N_3$.

4) $z \in \{O, \zeta_1, \zeta_2\}$.

In this case we have $\omega_n(0) = a_{n+1}\zeta_1\zeta_2$, whence $|\omega_n(0)| = |a_{n+1}| < \varepsilon_1$ for $n > N_1$, and $\omega_n(\zeta_{1,2}) = 0$.

Let $N = \max\{N_1, N_2, N_3\}$ and $n > N$, then having in view the inequalities (4.6) – (4.8), we can write on the boundary of the region Δ:

$$|\omega_n(z)| < \max\left(2C\varepsilon_1, (2RC+1)\varepsilon_1, \left(\frac{8CR^2}{\rho} + 1\right)\varepsilon_1\right) = \left(\frac{8CR^2}{\rho} + 1\right)\varepsilon_1.$$

Hence according to the principle of the maximum of the modulus

$$|\omega_n(z)| < \left(\frac{8CR^2 + \rho}{\rho}\right)\varepsilon_1 \tag{4.9}$$

on the arc σ.

Eventually, according to (2.4), (3.1) and (4.4), since $|z| = 1$ on the arc σ,

$$|\omega_n(z)| = \frac{\left|\psi(z) - \sum\limits_{k=0}^{n} a_k \widetilde{E}_{\alpha,k}(z)\right|}{|z^{n+1}||1+\theta_{\alpha,n+1}(z)|}|z-\zeta_1||z-\zeta_2| > \frac{1}{2}\cdot\frac{\rho^2}{4}\left|\psi(z) - \sum_{k=0}^{n} a_k \widetilde{E}_{\alpha,k}(z)\right|,$$

whence the inequality (4.9) yields

$$\left|\psi(z)-\sum_{k=0}^{n}a_k\widetilde{E}_{\alpha,k}(z)\right|<\frac{8}{\rho^2}|\omega_n(z)|<\frac{8\varepsilon_1}{\rho^3}\left(8CR^2+\rho\right)=\varepsilon.$$

The proofs for the other two series follow the same lines. ∎

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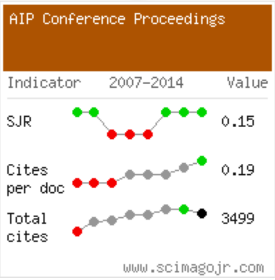

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